

Gravitational energy-momentum flow in binary systems

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Abstract

We investigate the gravitational energy-momentum distribution in the space-time of two black holes in circular orbit, in the context of the teleparallel equivalent of general relativity. This field configuration is important because gravitational waves are expected to be emitted in the final stages of inspiral and merger of binary black holes. We address an approximate solution of Einstein's field equations that describes two non-spinning black holes that circle each other in the xy plane, obtain the total energy of the space-time and verify that the gravitational binding energy is negative. We show that gravitational radiation is emitted as long as the separation between the holes decreases in time. If the black holes are spinning and circle each other, it has been found in the literature that, during the pre-merger inspiral, they bob up and down sinusoidally. The understanding of this phenomenon requires the understanding of the gravitational energy-momentum flow in the space-time of binary black holes. For the time dependent metric tensor of a general binary black hole system, the non-vanishing of the gravitational momentum may explain the bobbing of spinning black holes.

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1 Introduction

The physics of black holes coalescence is presently being intensively investigated. So far it is not known any exact solution of Einstein's field equations that describes the inspiral and merger of black holes. The progress in this field is due to post-Newtonian approximation methods and to nonlinear numerical simulations of the evolution of binary black hole space-times [1, 2, 3]. It is expected that binary black hole mergers will provide important information about the strong-field, nonlinear nature of the gravitational field, and will become a promising source of gravitational waves. Effects like radiation of mass (energy), linear and angular momentum are likely to take place in the final stages of black hole mergers. In particular, radiation of linear momentum by a binary black hole is related to the recoil of the final remnant hole (see refs. [4, 5, 6, 7] and references therein). If the remnant black hole acquires linear momentum, the latter should cancel with the linear momentum of the field in order to comply with the conservation of the total linear momentum of the configuration.

An intriguing and interesting phenomenon is the orbital motion of two identical, spinning black holes in quasi-circular orbit, with oppositely directed spins restricted to the orbital plane [5, 8, 9]. During the pre-merger inspiral the orbital plane of the binary black holes (plane xy , say) carry out a movement up and down along the z direction, i.e., the two black holes bob up and down sinusoidally and synchronously. After the merger the remnant black hole acquires a recoil velocity, and in realistic situations it may be ejected from the nucleus of the host galaxy [4, 5, 7].

The conservation of the total linear momentum of the space-time implies that the linear momentum of the field must be the same in magnitude, but opposite in sign, to the linear momentum of the binary/merged black holes [8]. Therefore the description of the linear momentum of the gravitational field in the space-time of binary black holes is important for the understanding of the physics of this configuration. Ideally one would like to know the details of the momentum flow between the fields and holes. These issues are mathematically intricate, but in principle they can be addressed (at least formally) in the framework of the teleparallel equivalent of general relativity (TEGR), provided a realistic post-Newtonian expression for the metric tensor is known. In the TEGR the expressions for the energy-momentum and angular momentum of the gravitational field are invariant under transformations of the coordinates of the three-dimensional spacelike surface, under

time reparametrizations, but depend on the frame of an observer. But normally the observer is stationary in the asymptotically flat space-time. The energy-momentum of any physical system in special relativity depends on the frame of an observer, and there is no special reason for dropping this feature when considering general relativity. The gravitational energy is the zero component of the gravitational energy-momentum four-vector, and thus it has standard transformation properties.

In this paper, we first address an approximate solution [10, 11] of Einstein's equations that describes two nonspinning black holes in circular motion, and evaluate the gravitational energy-momentum of the space-time. In this approximate model the orbital motion of the holes is restricted to the xy plane and the separation b between the holes is considered fixed (in the context of ref. [10]). For two black holes with individual rest masses m_1 and m_2 , we will find that the average value (in time) of the total energy $cP^{(0)}$ of the combined system in orbital motion is less than $c(P_1^{(0)} + P_2^{(0)}) = (m_1 + m_2)c^2$. The binding energy $E_b = cP^{(0)} - c(P_1^{(0)} + P_2^{(0)})$ is negative, its expression is very simple and is in agreement with previous analyses. Assuming that the separation b between the two black holes decreases in time, i.e., $\dot{b} < 0$, the binding energy yields a positive flux of gravitational radiation. For a fixed value of b we arrive at a simple expression for the total flux of gravitational radiation. We will show that the average value of the radiation over a period is zero. The conclusion is that effective gravitational radiation takes place provided $\dot{b} \neq 0$.

We also consider the general form of the metric tensor that describes the space-time of two spinning black holes in quasi-circular motion in the xy plane [8, 12], and evaluate the gravitational energy-momentum of the space-time. We obtain formal expressions for the linear momentum of the field along the x , y and z directions, contained within a large rectangular volume with sides a . This length is supposed to be much larger than the separation b between the two black holes. We find that, in general, the linear momenta vary with time. The dependence in time of the z component of the linear momentum is likely to be related to the bobbing of realistic binary black holes, prior to the merger. In our opinion, the present approach is better suited for this analysis, rather than the one based on pseudotensors [8]. Pseudotensors are quantities that depend on the choice of the coordinates of the three-dimensional space, and therefore they are not well behaved under coordinate transformations. The analysis developed in ref. [8] makes use of the Landau-

Lifshitz pseudotensor [13]. However, we have never seen a justification as to why one pseudotensor is better than another one.

This paper is organized as follows. In section 2, we review the formulation of the TEGR, and show how the definition of the gravitational energy-momentum arises out of the field equations of the theory. We also present the expression for the fluxes of gravitational radiation and radiation of matter fields. In section 3, we describe the approximate solution for two black holes in circular orbit, and in section 4 we evaluate the energy and momentum of the gravitational field. We find simple expressions for the total gravitational energy, for the binding energy and for the total flux of gravitational radiation. In section 5, we consider the general form of the metric tensor for the inspiral of two spinning black holes that circle each other, and obtain the formal expressions for the gravitational momenta along the three spatial directions. Assuming the standard asymptotic behaviour of the metric tensor components, it will be clear that the momentum components of the gravitational field are time dependent, a fact that very likely explains the bobbing of the black holes. Finally we present our conclusions in section 6.

Notation: space-time indices μ, ν, \dots and $\text{SO}(3,1)$ indices a, b, \dots run from 0 to 3. Time and space indices are indicated according to $\mu = 0, i, \quad a = (0), (i)$. The tetrad field is denoted $e^a{}_\mu$, and the torsion tensor reads $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$. The flat, Minkowski spacetime metric tensor raises and lowers tetrad indices and is fixed by $\eta_{ab} = e_{a\mu} e_{b\nu} g^{\mu\nu} = (-1, +1, +1, +1)$. The determinant of the tetrad field is represented by $e = \det(e^a{}_\mu)$.

2 Energy-momentum in the TEGR

In the teleparallel equivalent of general relativity the gravitational field is represented by the tetrad field $e^a{}_\mu$ only, and the Lagrangian density is written in terms of the torsion tensor $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$. This tensor is related to the antisymmetric part of the Weitzenböck connection $\Gamma_{\mu\nu}^\lambda = e^{a\lambda} \partial_\mu e_{a\nu}$. However, the dynamics of the gravitational field in the TEGR is essentially the same as in the usual metric formulation. The physics in both formulations is identically the same.

Let us start with the torsion-free, Levi-Civita connection ${}^0\omega_{\mu ab}$,

$$\begin{aligned}
{}^0\omega_{\mu ab} &= -\frac{1}{2}e^c{}_\mu(\Omega_{abc} - \Omega_{bac} - \Omega_{cab}), \\
\Omega_{abc} &= e_{a\nu}(e_b{}^\mu\partial_\mu e_c{}^\nu - e_c{}^\mu\partial_\mu e_b{}^\nu).
\end{aligned} \tag{1}$$

The Christoffel symbols ${}^0\Gamma_{\mu\nu}^\lambda$ and the Levi-Civita connection are identically related by

$${}^0\Gamma_{\mu\nu}^\lambda = e^{a\lambda}\partial_\mu e_{a\nu} + e^{a\lambda}({}^0\omega_{\mu ab})e^b{}_\nu.$$

In view of this expression an identity arises between the Levi-Civita connection and the contorsion tensor $K_{\mu ab}$,

$${}^0\omega_{\mu ab} = -K_{\mu ab}, \tag{2}$$

where $K_{\mu ab} = \frac{1}{2}e_a{}^\lambda e_b{}^\nu(T_{\lambda\mu\nu} + T_{\nu\lambda\mu} + T_{\mu\lambda\nu})$, and $T_{\lambda\mu\nu} = e^a{}_\lambda T_{a\mu\nu}$. Making use of eq. (2) it follows that the scalar curvature $R(e)$ may be identically written as

$$eR({}^0\omega) = -e\left(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^a T_a\right) + 2\partial_\mu(eT^\mu), \tag{3}$$

where e is the determinant of the tetrad field. Therefore in the framework of the TEGR the Lagrangian density for the gravitational and matter fields is defined by

$$\begin{aligned}
L &= -ke\left(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^a T_a\right) - \frac{1}{c}L_M \\
&\equiv -ke\Sigma^{abc}T_{abc} - \frac{1}{c}L_M,
\end{aligned} \tag{4}$$

where $k = c^3/16\pi G$, $T_a = T^b{}_{ba}$, $T_{abc} = e_b{}^\mu e_c{}^\nu T_{a\mu\nu}$ and

$$\Sigma^{abc} = \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac}T^b - \eta^{ab}T^c). \tag{5}$$

L_M stands for the Lagrangian density for the matter fields. The Lagrangian density L is invariant under the global $SO(3,1)$ group. The absence in the Lagrangian density of the divergence term on the right hand side of eq. (3) prevents the invariance of (4) under arbitrary local $SO(3,1)$ transformations.

The field equations derived from (4) are equivalent to Einstein's equations. They read

$$e_{a\lambda}e_{b\mu}\partial_\nu(e\Sigma^{b\lambda\nu}) - e(\Sigma^{b\nu}{}_a T_{b\nu\mu} - \frac{1}{4}e_{a\mu}T_{bcd}\Sigma^{bcd}) = \frac{1}{4kc}eT_{a\mu}, \quad (6)$$

where $\delta L_M/\delta e^{a\mu} = eT_{a\mu}$. From now on we will make $c = 1 = G$.

The definition of the gravitational energy-momentum may be established in the framework of the Lagrangian formulation defined by (4), according to the procedure of ref. [14]. Equation (6) may be rewritten as

$$\partial_\nu(e\Sigma^{a\lambda\nu}) = \frac{1}{4k}e e^a{}_\mu(t^{\lambda\mu} + T^{\lambda\mu}), \quad (7)$$

where $T^{\lambda\mu} = e_a{}^\lambda T^{a\mu}$ and $t^{\lambda\mu}$ is defined by

$$t^{\lambda\mu} = k(4\Sigma^{bc\lambda}T_{bc}{}^\mu - g^{\lambda\mu}\Sigma^{bcd}T_{bcd}). \quad (8)$$

In view of the antisymmetry property $\Sigma^{a\mu\nu} = -\Sigma^{a\nu\mu}$ it follows that

$$\partial_\lambda [e e^a{}_\mu(t^{\lambda\mu} + T^{\lambda\mu})] = 0. \quad (9)$$

The equation above yields the continuity (or balance) equation,

$$\frac{d}{dt} \int_V d^3x e e^a{}_\mu(t^{0\mu} + T^{0\mu}) = - \oint_S dS_j [e e^a{}_\mu(t^{j\mu} + T^{j\mu})], \quad (10)$$

Therefore we identify $t^{\lambda\mu}$ as the gravitational energy-momentum tensor [14],

$$P^a = \int_V d^3x e e^a{}_\mu(t^{0\mu} + T^{0\mu}), \quad (11)$$

as the total energy-momentum contained within a volume V of the three-dimensional space,

$$\Phi_g^a = \oint_S dS_j (e e^a{}_\mu t^{j\mu}), \quad (12)$$

as the gravitational energy-momentum flux [15], and

$$\Phi_m^a = \oint_S dS_j (e e^a{}_\mu T^{j\mu}), \quad (13)$$

as the energy-momentum flux of matter. In view of (7) eq. (11) may be written as

$$P^a = - \int_V d^3x \partial_j \Pi^{aj} = - \oint_S dS_j \Pi^{aj}, \quad (14)$$

where $\Pi^{aj} = -4ke \Sigma^{a0j}$. The expression above is the definition for the gravitational energy-momentum presented in ref. [16], obtained in the framework of the vacuum field equations in Hamiltonian form. It is invariant under coordinate transformations of the three-dimensional space and under time reparametrizations. We note that (9) is a true energy-momentum conservation equation.

Finally we remark that in the absence of matter fields the total flux of gravitational radiation $\Phi^{(0)}$ is related to the total gravitational energy according to $\Phi^{(0)} = -\dot{P}^{(0)}$, in view of eq. (10).

3 The space-time of the binary black hole

We consider the approximate metric for the binary black hole as described in ref. [10]. This solution was later re-analyzed and re-obtained in ref. [11] by matching two perturbed Schwarzschild metrics to an asymptotically post-Newtonian construction for a binary black hole space-time. The two black holes have masses m_1 and m_2 , and circle around each other in the plane xy . We restrict the analysis to the coordinates in the radiation zone, defined by $r \gg \lambda_{GW}$ [17, 18], where λ_{GW} is the wavelength of the gravitational radiation. In the context of the present analysis the radiation zone is established in an equivalent way by $r \gg m_1$, $r \gg m_2$ and $r \gg b$ (see below the definition of b). We define

$$m = m_1 + m_2, \quad \delta m = m_1 - m_2, \quad \mu = \frac{m_1 m_2}{m}. \quad (15)$$

The circular, Newtonian trajectories of the black holes are

$$\mathbf{r}_1(t) = \frac{m_2}{m} \mathbf{b}(t), \quad \mathbf{r}_2(t) = -\frac{m_1}{m} \mathbf{b}(t), \quad (16)$$

where

$$\mathbf{b}(t) = \mathbf{r}_1(t) - \mathbf{r}_2(t) = b(\cos \omega t, \sin \omega t, 0), \quad (17)$$

and

$$\omega = \sqrt{\frac{m}{b^3}},$$

is the orbital angular velocity. The separation b is defined by $b = |\mathbf{r}_1 - \mathbf{r}_2|$. The velocity of the holes are

$$\mathbf{v}_1 = \frac{d\mathbf{r}_1}{dt}, \quad \mathbf{v}_2 = \frac{d\mathbf{r}_2}{dt}, \quad (18)$$

from what we define

$$\mathbf{v}(t) = \mathbf{v}_1 - \mathbf{v}_2 = \sqrt{\frac{\tilde{m}}{b}}(-\sin \omega t, \cos \omega t, 0). \quad (19)$$

In the radiation zone the metric components depend on \mathbf{b} and \mathbf{v} , which in turn depend not exactly on t , but on the retarded time $\tau = t - r$ [10]. We further define

$$\tilde{m} = m \left(1 - \frac{\mu}{2b}\right), \quad \mathbf{n} = \frac{\mathbf{r}}{r}, \quad (20)$$

where \mathbf{r} is a point of observation in space.

With the help of the definitions above the metric tensor components for the binary black hole space-time in the radiation zone read [10]

$$\begin{aligned} g_{00} &= -1 + \frac{2\tilde{m}}{r} - \frac{2\tilde{m}^2}{r^2} \\ &\quad + \frac{\mu}{r} \left\{ 2(\mathbf{n} \cdot \mathbf{v})^2 - \frac{2m}{b^3}(\mathbf{n} \cdot \mathbf{b})^2 + \frac{6}{r}(\mathbf{n} \cdot \mathbf{b})(\mathbf{n} \cdot \mathbf{v}) + \frac{1}{r^2}[3(\mathbf{n} \cdot \mathbf{b})^2 - b^2] \right\} \\ &\quad + \frac{\mu}{r} \frac{\delta m}{m} \left\{ (\mathbf{n} \cdot \mathbf{v}) \left[\frac{7m}{b^3}(\mathbf{n} \cdot \mathbf{b})^2 - 2(\mathbf{n} \cdot \mathbf{v})^2 - \frac{m}{b} \right] \right. \\ &\quad + \frac{2}{r}(\mathbf{n} \cdot \mathbf{b}) \left[\frac{3m}{b^3}(\mathbf{n} \cdot \mathbf{b})^2 - 6(\mathbf{n} \cdot \mathbf{v})^2 - \frac{m}{b} \right] \\ &\quad \left. + \frac{3}{r^2}(\mathbf{n} \cdot \mathbf{v})[b^2 - 5(\mathbf{n} \cdot \mathbf{b})^2] + \frac{1}{r^3}(\mathbf{n} \cdot \mathbf{b})[3b^2 - 5(\mathbf{n} \cdot \mathbf{b})^2] \right\}, \\ g_{0i} &= -\frac{4\mu}{r} \left\{ \left[(\mathbf{n} \cdot \mathbf{v}) + \frac{1}{r}(\mathbf{n} \cdot \mathbf{b}) \right] v^i - \frac{m}{b^3}(\mathbf{n} \cdot \mathbf{b})b^i \right\} \\ &\quad + \frac{2\mu}{r} \frac{\delta m}{m} \left\{ \left(2(\mathbf{n} \cdot \mathbf{v})^2 - \frac{3m}{b^3}(\mathbf{n} \cdot \mathbf{b})^2 + \frac{6}{r}(\mathbf{n} \cdot \mathbf{b})(\mathbf{n} \cdot \mathbf{v}) \right. \right. \\ &\quad \left. \left. + \frac{1}{r^2}[3(\mathbf{n} \cdot \mathbf{b})^2 - b^2] \right) v^i \right. \\ &\quad \left. + \left\{ -\frac{4m}{b^3}(\mathbf{n} \cdot \mathbf{b})(\mathbf{n} \cdot \mathbf{v}) + \frac{m}{rb} \left[1 - \frac{3}{b^2}(\mathbf{n} \cdot \mathbf{b})^2 \right] \right\} b^i \right\}, \\ g_{ij} &= \delta_{ij} \left(1 + \frac{2\tilde{m}}{r} + \frac{m^2}{r^2} \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{\mu}{r} \left\{ 2(\mathbf{n} \cdot \mathbf{v})^2 - \frac{2m}{b^3}(\mathbf{n} \cdot \mathbf{b})^2 + \frac{6}{r}(\mathbf{n} \cdot \mathbf{b})(\mathbf{n} \cdot \mathbf{v}) + \frac{1}{r^2}[3(\mathbf{n} \cdot \mathbf{b})^2 - b^2] \right\} \\
& + \frac{\mu}{r} \frac{\delta m}{m} \left\{ (\mathbf{n} \cdot \mathbf{v}) \left[\frac{7m}{b^3}(\mathbf{n} \cdot \mathbf{b})^2 - 2(\mathbf{n} \cdot \mathbf{v})^2 + \frac{m}{b} \right] \right. \\
& + \frac{6}{r}(\mathbf{n} \cdot \mathbf{b}) \left[\frac{m}{b^3}(\mathbf{n} \cdot \mathbf{b})^2 - 2(\mathbf{n} \cdot \mathbf{v})^2 \right] \\
& + \frac{3}{r^2}(\mathbf{n} \cdot \mathbf{v})[b^2 - 5(\mathbf{n} \cdot \mathbf{b})^2] + \frac{1}{r^3}(\mathbf{n} \cdot \mathbf{b})[3b^2 - 5(\mathbf{n} \cdot \mathbf{b})^2] \left. \right\} \\
& + \frac{m^2}{r^2} n^i n^j + \frac{4\mu}{r} \left(v^i v^j - \frac{m}{b^3} b^i b^j \right) + \frac{2\mu}{r} \frac{\delta m}{m} \left\{ \frac{6m}{b^3}(\mathbf{n} \cdot \mathbf{b}) v^{(i} b^{j)} \right. \\
& + \left. \left[(\mathbf{n} \cdot \mathbf{v}) + \frac{1}{r}(\mathbf{n} \cdot \mathbf{b}) \right] \left(\frac{m}{b^3} b^i b^j - 2v^i v^j \right) \right\}. \tag{21}
\end{aligned}$$

Our aim is to evaluate definitions (14) and (12) for the gravitational energy-momentum and the corresponding fluxes. These definitions are invariant under global SO(3,1) transformations. Therefore they are frame dependent. However, P^a is a vector under global Lorentz transformations. The frame dependence of the gravitational energy-momentum is understood by simply considering a black hole of mass m and an observer that is very distant from the black hole. The black hole will appear to this observer as a particle of mass m , with energy $E = cP^{(0)} = mc^2$ (m is the mass of the black hole in the frame where the black hole is at rest). If, however, the black hole is moving at velocity v with respect to the observer, then its total gravitational energy will be $E = \gamma mc^2$, where $\gamma = (1 - v^2/c^2)^{-1/2}$. This example is a consequence of the special theory of relativity, and shows clearly the frame dependence of the gravitational energy-momentum. The frame dependence is not restricted to observers at spacelike infinity. It holds for observers located everywhere in the three-dimensional space.

In order to evaluate definitions (11-14) out of the metric tensor given by (21) we choose a configuration of tetrad fields that has a clear physical interpretation. In the framework of the TEGR the tetrad field describes both the gravitational field and the frame. For a given metric tensor there exists an infinity of possible frames, and each frame is characterized by six conditions on the tetrad field. Three conditions fix the kinematical state of the observer in the three-dimensional space (for instance, the observer may be stationary in space), and the other three conditions fix the orientation of the frame (alternatively, the frame may be characterized by the six components of the acceleration tensor ϕ_{ab} [19]).

Therefore tetrad fields are interpreted as reference frames adapted to preferred fields of observers in spacetime. This interpretation is possible by identifying the $e_{(0)}{}^\mu$ components of the frame with the four-velocities u^μ of the observers, $e_{(0)}{}^\mu = u^\mu$ [19]. Here we will establish a set of tetrad fields adapted to static observers in spacetime. Thus we require $e_{(0)}{}^i = 0$. This condition fixes 3 components of the frame. The other three components are fixed by choosing an orientation of the frame in the three-dimensional space. Therefore $e_{(0)}{}^\mu$ is parallel to the worldline of the observers, and $e_{(k)}{}^\mu$ are the three unit vectors orthogonal to the timelike direction. We fix $e_{(k)}{}^\mu$ such that $e_{(1)}{}^\mu$, $e_{(2)}{}^\mu$ and $e_{(3)}{}^\mu$ in cartesian coordinates (and in the flat space-time limit) are unit vectors along the x , y and z directions. The tetrad field that satisfies these conditions is given by

$$e^a{}_\mu(t, x, y, z) = \begin{pmatrix} A & B & C & 0 \\ 0 & D & E & F \\ 0 & 0 & G & H \\ 0 & 0 & 0 & I \end{pmatrix}, \quad (22)$$

with the following definitions:

$$\begin{aligned} A &= (-g_{00})^{1/2}, \\ B &= -\frac{g_{01}}{(-g_{00})^{1/2}}, \\ C &= -\frac{g_{02}}{(-g_{00})^{1/2}}, \\ D &= \frac{\lambda_{11}}{(-g_{00})^{1/2}}, \\ E &= \frac{1}{(-g_{00})^{1/2}} \frac{\lambda_{12}^2}{\lambda_{11}}, \\ F &= (-g_{00})^{1/2} \frac{g_{13}}{\lambda_{11}}, \\ G &= \frac{1}{(-g_{00})^{1/2}} \left[\lambda_{22}^2 - \frac{\lambda_{12}^4}{\lambda_{11}^2} \right]^{1/2}, \\ H &= \frac{(-g_{00})^{1/2}}{\lambda_{11}} \frac{g_{23}\lambda_{11}^2 - g_{13}\lambda_{12}^2}{(\lambda_{11}^2\lambda_{22}^2 - \lambda_{12}^4)^{1/2}}, \\ I &= \frac{1}{\lambda_{11}} \left[g_{33}\lambda_{11}^2 - (-g_{00}) \left(g_{13}^2 + \frac{(g_{23}\lambda_{11}^2 - g_{13}\lambda_{12}^2)^2}{(\lambda_{11}^2\lambda_{22}^2 - \lambda_{12}^4)} \right) \right]^{1/2}. \end{aligned} \quad (23)$$

The quantity λ_{ij} is defined by $\lambda_{ij}^2 = g_{0i}g_{0j} - g_{00}g_{ij}$, and all metric components are obtained from (21). In the limit $r \rightarrow \infty$ the asymptotic quantities h_{00}, h_{11}, h_{22} and h_{33} are defined by the expressions

$$\begin{aligned} g_{00} &= -1 + h_{00} , \\ g_{11} &= 1 + h_{11} , \\ g_{22} &= 1 + h_{22} , \\ g_{33} &= 1 + h_{33} . \end{aligned} \tag{24}$$

In terms of these quantities the asymptotic form of the tetrad field is reduced to

$$e^a{}_\mu(t, x, y, z) \cong \begin{pmatrix} 1 - \frac{h_{00}}{2} & -g_{01} & -g_{02} & 0 \\ 0 & 1 + \frac{h_{11}}{2} & g_{12} & g_{13} \\ 0 & 0 & 1 + \frac{h_{22}}{2} & g_{23} \\ 0 & 0 & 0 & 1 + \frac{h_{33}}{2} \end{pmatrix} . \tag{25}$$

Expression (22) represents a frame that is adapted to static observers everywhere in space-time.

4 Gravitational energy of binary black holes in circular motion

For a given space-time metric tensor and a given frame, the energy-momentum of the space-time is evaluated out of eq. (14). It reads

$$P^a = 4k \oint_S dS_j e \Sigma^{a0j} . \tag{26}$$

If the surface of integration S is fixed at spatial infinity, i.e., $S \rightarrow \infty$, P^a yields the total energy-momentum of the space-time. The latter is the same for all tetrad fields that exhibit the same asymptotic behaviour. In particular, the energy-momentum obtained out of frames that are adapted to static observers at spacelike infinity coincides with the one obtained out of (22).

Considering the tetrad field given by eq. (22), the gravitational energy of the space-time determined by eq. (21) is given by

$$\begin{aligned}
P^{(0)} &= 4ke \oint_S dS_j e(e^{(0)}{}_0 \Sigma^{00j} + e^{(0)}{}_1 \Sigma^{10j} + e^{(0)}{}_2 \Sigma^{20j}) \\
&= 4k \oint_S dS_j e(A \Sigma^{00j} + B \Sigma^{10j} + C \Sigma^{20j}), \tag{27}
\end{aligned}$$

where A , B and C are defined by (23), $\Sigma^{\mu\nu\lambda}$ is calculated out of (5), and $j = 1, 2, 3$. We will evaluate the expression of $P^{(0)}$ for a closed surface S in the asymptotic region $r \gg m$ and $r \gg b$, which characterizes the radiation zone, and then we make $S \rightarrow \infty$. The metric tensor may be decomposed as $g_{\mu\nu} \cong \eta_{\mu\nu} + h_{\mu\nu}$, and $h_{\mu\nu}$ is of order $1/r$ at spacelike infinity. Thus we also have $g^{\mu\nu} \cong \eta^{\mu\nu} - h^{\mu\nu}$, where $h^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}$. In view of the intricate structure of the metric tensor given by (21), in the evaluation of (27) we will make the approximation $\Sigma^{\mu\nu\lambda} \cong \eta^{\mu\alpha} \eta^{\nu\beta} \eta^{\lambda\gamma} \Sigma_{\alpha\beta\gamma}$, making sure that $e \Sigma^{a0j}$ is of order (at least) $1/r^2$. After a large number of calculations, and taking into account the approximations explained above, we obtain the exact expression for the total energy of the space-time,

$$\begin{aligned}
P^{(0)} &= (m_1 + m_2) \left[1 - \frac{m_1 m_2}{2(m_1 + m_2)b} \right] \\
&- \left\{ \frac{8}{3} \frac{\mu^2 m^2}{b^2} \omega \cos(2\omega\tau) + \frac{\mu^2 (\delta m)^2}{b^2} \left(\frac{m\omega}{b} \right) \left[\frac{3}{2} \cos(2\omega\tau) \right. \right. \\
&\left. \left. + \frac{3}{8} \cos^4 \omega\tau + \frac{29}{8} \sin^4 \omega\tau \right] \right\} \sin \omega\tau \cos \omega\tau. \tag{28}
\end{aligned}$$

Assuming that the energy-momentum of matter fields vanishes for the binary black holes, then the expression above does represent the gravitational energy of the space-time.

We will be interested in average values of time dependent quantities. Therefore we define

$$\langle P^{(0)} \rangle = \frac{1}{T} \int_0^T d\tau P^{(0)}, \tag{29}$$

where $T = 2\pi/\omega$. We easily obtain

$$\langle P^{(0)} \rangle = (m_1 + m_2) \left[1 - \frac{m_1 m_2}{2(m_1 + m_2)b} \right]. \tag{30}$$

We define the binding energy of the configuration according to $E_b = \langle P^{(0)} \rangle - (m_1 + m_2)$. We find

$$E_b = -\frac{m_1 m_2}{2b}. \quad (31)$$

We note that E_b is the the standard non-spinning part of the expression for the binding gravitational energy [20, 21] for two black holes in circular orbit. This term is precisely the same as the first term in eq. (191) of ref. [22].

We evaluate now the total flux $\Phi^{(0)} = -\dot{P}^{(0)}$ of gravitational radiation. After a large number of calculations we arrive at

$$\begin{aligned} \Phi^{(0)} = & \left\{ \frac{8}{3} \frac{\mu^2 m^2}{b^2} \omega^2 \cos(2\omega\tau) + \frac{\mu^2 (\delta m)^2}{b^2} \left(\frac{m\omega^2}{b} \right) \left[\frac{3}{2} \cos(2\omega\tau) + \frac{3}{8} \cos^4 \omega\tau \right. \right. \\ & + \left. \left. \frac{29}{8} \sin^4 \omega\tau \right] \right\} \cos(2\omega\tau) \\ & + \left\{ -\frac{8}{3} \frac{\mu^2 m^2}{b^2} (2\omega^2) \sin(2\omega\tau) + \frac{\mu^2 (\delta m)^2}{b^2} \left(\frac{m\omega^2}{b} \right) \left[-3 \sin(2\omega\tau) \right. \right. \\ & - \left. \left. \frac{3}{2} \cos^3 \omega\tau \sin \omega\tau + \frac{29}{2} \sin^3 \omega\tau \cos \omega\tau \right] \right\} \sin \omega\tau \cos \omega\tau. \end{aligned} \quad (32)$$

An interesting consequence of the expression above is that the average value of $\Phi^{(0)}$ over a complete cycle vanishes,

$$\langle \Phi^{(0)} \rangle = 0. \quad (33)$$

We conclude that the orbital (stationary) motion of two black holes (or two point masses) on a plane produce the gravitational radiation given by eq. (32). However, the average value of this radiation vanishes. On the other hand, the situation changes if the separation distance b changes with time.

In order to obtain (32) we have considered that b is not a function of t . This condition was assumed in ref. [10]. According to ref. [11], however, this condition may be relaxed. Thus we may admit that b varies slowly with time. Taking into account this assumption we obtain an expression for the total flux of the gravitational radiation. For arbitrary $b(t)$ we have $\dot{\omega} \neq 0$. Consequently neither a complete cycle nor a period can be defined. In fact, the concept of average value does not apply to this case. $\Phi^{(0)}$ is now clearly nonvanishing. We present here only the contribution to $\Phi^{(0)}$ of the variation in time of the first term on the right hand side of eq. (28), namely, the variation in time of the binding energy, which does not even contribute to (32). We find

$$\Phi^{(0)} \approx -\frac{dE_b}{dt} = -\frac{m_1 m_2}{2b^2} \dot{b}. \quad (34)$$

This flux will be positive definite provided $\dot{b} < 0$, which is the case for realistic binary black holes prior to the merger.

We have evaluated the momentum component $P^{(3)}$, which is the component oriented along the z direction. The integration is carried out over a finite cubic volume with sides a , such that $a \gg b$. We found that $P^{(3)} = 0$. This result is expected since the black holes are restricted to the xy plane, and there is no flux of momentum along the z direction, in contrast to a general situation to be addressed in the next section. When the integration is carried out over the whole three-dimensional space, the total momentum of the space-time vanishes, i.e., $P^{(i)} = 0$ for $i = 1, 2, 3$.

5 Arbitrary time dependent metric tensor

The standard form of the metric tensor for the space-time of the inspiral of two black holes is given by [23, 12, 8]

$$\begin{aligned} g_{00} &= -1 + 2V - 2V^2 + 8X \\ g_{0i} &= -4V_i - 8R_i \\ g_{ij} &= \delta_{ij}(1 + 2V + 2V^2) + 4W_{ij}. \end{aligned} \quad (35)$$

The form of the potentials V, V_i, X, R_i and W_{ij} may be obtained in the three references indicated above. Here we will just assume that V, V_i, R_i and X behave as $1/r$ at spacelike infinity, whereas W_{ij} behaves as $1/r^2$. The expression of the energy-momentum P^a will be given in terms of these potentials. The use of the explicit form of the potentials yields a rather intricate form of the energy-momentum. The spinning nature of the solution is manifest in the potentials V, V_i and W_{ij} [12].

In cartesian coordinates a stationary observer in space-time is described by a frame very similar to eq. (22). In the metric tensor given by (21) we have $g_{03} = 0$, which is not the case for (35). The frame that (i) yields (35), (ii) is adapted to stationary observers in space-time, i.e., $e_{(0)}{}^i = 0$, and (iii) is oriented along the x, y, z directions at spacelike infinity, i.e., $e_{(i)}{}^j(t, x, y, z) \cong \delta_i^j$ when $r \rightarrow \infty$, is given by

$$e^a{}_\mu = \begin{pmatrix} A & B & C & J \\ 0 & D & E & F \\ 0 & 0 & G & H \\ 0 & 0 & 0 & I \end{pmatrix}, \quad (36)$$

where the following relations are satisfied,

$$\begin{aligned} A^2 &= -g_{00} \\ AB &= -g_{01} \\ AC &= -g_{02} \\ AJ &= -g_{03} \\ -B^2 + D^2 &= g_{11} \\ -BC + DE &= g_{12} \\ -BJ + FD &= g_{13} \\ -C^2 + E^2 + G^2 &= g_{22} \\ -CJ + EF + GH &= g_{23} \\ -J^2 + F^2 + H^2 + I^2 &= g_{33}. \end{aligned} \quad (37)$$

The relations above allow to obtain all tetrad components in terms of the metric tensor components.

We present below all components of the energy-momentum obtained out of the tetrad field (36), assuming the asymptotic behaviour of the potentials as explained above. We have discarded several terms that fall off as $O(1/r^3)$ or faster. For $P^{(0)}$ we obtain

$$\begin{aligned} P^{(0)} &= -k \left\{ \lim_{x \rightarrow \pm\infty} \int_{-\infty}^{\infty} dydz [\partial_1(g_{22} + g_{33})] \right. \\ &+ \lim_{y \rightarrow \pm\infty} \int_{-\infty}^{\infty} dx dz [\partial_2(g_{11} + g_{33})] \\ &+ \lim_{z \rightarrow \pm\infty} \int_{-\infty}^{\infty} dx dy [\partial_3(g_{11} + g_{22})] \\ &+ \lim_{x \rightarrow \pm\infty} \int_{-\infty}^{\infty} dydz [C\partial_0(g_{12}) + J\partial_0(g_{13})] \\ &+ \lim_{y \rightarrow \pm\infty} \int_{-\infty}^{\infty} dx dz [B\partial_0(g_{12}) + J\partial_0(g_{23})] \end{aligned}$$

$$\begin{aligned}
& + \lim_{z \rightarrow \pm\infty} \int_{-\infty}^{\infty} dx dy [B \partial_0(g_{13}) + C \partial_0(g_{23})] \\
& + \lim_{x \rightarrow \pm\infty} \int_{-\infty}^{\infty} dy dz B \partial_0(g_{11}) + \lim_{y \rightarrow \pm\infty} \int_{-\infty}^{\infty} dx dz C \partial_0(g_{22}) \\
& + \lim_{z \rightarrow \pm\infty} \int_{-\infty}^{\infty} dx dy J \partial_0(g_{33}) \}, \tag{38}
\end{aligned}$$

where

$$\lim_{x \rightarrow \pm\infty} \int_{-\infty}^{\infty} dy dz F(x, y, z) = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz F(\infty, y, z) - \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz F(-\infty, y, z).$$

Equation (38) reproduces (28) if we reduce the metric tensor (35) to the form given by (21), in which case we make $J = 0$.

In view of the asymptotic behaviour of the metric tensor (35), the total momentum of the space-time vanishes. In the expressions below for $P^{(i)}$, we formally integrate over a finite surface S_0 of a large rectangular volume with sides $(2x_0, 2y_0, 2z_0)$, such that $x_0 = y_0 = z_0 = a$, and a is much larger than the separation of the black holes. We arrive at

$$\begin{aligned}
P^{(1)} &= -32k \left[\int_{S_0} dy dz (V_1 \partial_0 V_1) + \int_{S_0} dz dx (V_1 \partial_0 V_2) \right. \\
&+ \left. \int_{S_0} dx dy (V_1 \partial_0 V_3) \right] \\
&+ 4k \left[\int_{S_0} dy dz \partial_0 (V + W_{22} + W_{33}) \right. \\
&- \left. \int_{S_0} dx dy (\partial_0 W_{13}) - \int_{S_0} dz dx (\partial_0 W_{12}) \right], \tag{39}
\end{aligned}$$

$$\begin{aligned}
P^{(2)} &= -32k \left[\int_{S_0} dy dz (V_2 \partial_0 V_1) + \int_{S_0} dz dx (V_2 \partial_0 V_2) \right. \\
&+ \left. \int_{S_0} dx dy (V_2 \partial_0 V_3) \right] \\
&+ 4k \left[\int_{S_0} dz dx \partial_0 (V + W_{11} + W_{33}) \right. \\
&- \left. \int_{S_0} dy dz (\partial_0 W_{12}) - \int_{S_0} dx dy (\partial_0 W_{23}) \right], \tag{40}
\end{aligned}$$

$$\begin{aligned}
P^{(3)} &= -32k \left[\int_{S_0} dy dz (V_3 \partial_0 V_1) + \int_{S_0} dz dx (V_3 \partial_0 V_2) \right. \\
&\quad \left. + \int_{S_0} dx dy (V_3 \partial_0 V_3) \right] \\
&\quad + 4k \left[\int_{S_0} dy dz \partial_0 (V + W_{11} + W_{22}) \right. \\
&\quad \left. - \int_{S_0} dz dx (\partial_0 W_{23}) - \int_{S_0} dy dz (\partial_0 W_{13}) \right], \tag{41}
\end{aligned}$$

where

$$\int_{S_0} dy dz F(x, y, z) = \int_{-y_0}^{y_0} dy \int_{-z_0}^{z_0} dz F(x_0, y, z) - \int_{-y_0}^{y_0} dy \int_{-z_0}^{z_0} dz F(-x_0, y, z),$$

etc. An important conclusion that we can draw from the expressions above is that the potentials X and R_i do not contribute to the momenta.

Considering the post-Newtonian potentials V , V_i and W_{ij} , we find that the $P^{(3)}$ component of the gravitational momentum is, in general, nonvanishing for a finite volume of the three-dimensional space, and exhibits a dependence in time. Therefore, it yields a momentum flux $\phi^{(3)} = -\dot{P}^{(3)}$ which, in turn, is likely to be related to the bobbing of the black holes. Unfortunately the post-Newtonian potentials of ref. [23] are not suitable for the present analysis, because they are valid only in the near zone, and we are interested in the expressions of the potentials in the radiation zone. Some of the potentials presented in the latter reference diverge with the increasing of the radial distance r , a feature that prevents us from calculating all momentum components.

The post-Newtonian potentials V , V_i and W_{ij} depend on the spins S_x , S_y and S_z of the black holes, which are time dependent functions [12]. The explicit form of these functions, for given initial conditions, and an analytic, widely accepted form of the post-Newtonian potentials, are not available in the literature. For this reason, we cannot proceed and obtain the detailed form of expressions (39), (40) and (41).

6 Concluding remarks

In this paper we have analyzed the metric tensor for the nonspinning black hole binary in circular orbit in the xy plane, in the context of the teleparallel equivalent of general relativity. This metric tensor is an approximate solution of Einstein's equations in which the distance between the holes is constant in time.

We have also addressed the general post-Newtonian form of the metric tensor that describes the inspiral and merger of two spinning black holes. The total energy-momentum of the space-time may be expressed in a simple form in terms of the metric tensor components, and may be easily computed provided the post-Newtonian potentials are given. We have found that only the potentials V , V_i and W_{ij} contribute to the momenta. It is very likely that the time dependence of the momentum component $P^{(3)}$ (the momentum component oriented along the z direction) is related to the bobbing of the spinning black holes with oppositely directed spins restricted to the orbital plane.

The calculations of the total gravitational energy, and the corresponding flux for the nonspinning black hole binary in circular orbit, yield a quite interesting result. The total energy $P^{(0)}$ and the gravitational flux $\phi^{(0)}$ are given by eqs. (28) and (32). The former yields the known result for the mean value of the binding energy of the configuration, whereas the average value of $\phi^{(0)}$ in time vanishes, $\langle \phi^{(0)} \rangle = 0$. It means that for two black holes in circular orbit, the average value of the gravitational radiation is zero. This result is consistent with the stationary character of the space-time. A nontrivial emission of gravitational radiation must necessarily be related to a loss of energy-momentum of the source (as the loss of mass described by Bondi's radiating metric [24]). If, however, the separation between the holes decreases in time, as in an actual evolution of the black hole binary, a nonvanishing (definite positive) flux of gravitational radiation is emitted.

This result is conceptually different from the conclusion drawn from Eddington's spinning rod [25], which was reconsidered by other authors [26]. In the framework of pseudotensor definitions and of linearized general relativity, the quadrupole formula was obtained. The latter relates the energy loss of the system with the variation in time of the mass-quadrupole of the source. This formula was used by Eddington to deduce the energy flux generated by a rod that spins in the xy plane with angular frequency ω . Let I represent the moment of inertia of the rod, and G the gravitational constant. The total

energy flux is given by [25, 13]

$$\frac{dE}{dt} = -\frac{32GI^2\omega^6}{5c^5}. \quad (42)$$

In contrast to the approach that allows to deduce the formula above, we note that the procedure that led to eqs. (32) and (33) is based neither on pseudotensors nor on the linearized form of the theory. Equation (10) is a true tensorial quantity. It is valid for any coordinate system of the three-dimensional spacelike hypersurface, and for finite volume V and corresponding surface S . We believe that the reconsideration of Eddington's spinning rod in the present context would also lead to an equation of the type $\langle \phi^{(0)} \rangle = 0$, provided the angular frequency is constant.

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